18/10/23

MATH2050A Tutorial

Reminders:

- HW3 due 23/10 on Gradescope.
- Midterm 2 on 15/11
- Return graded midterms by (aim for) next week.

Q2: a) Let $A, B \subseteq \mathbb{R}$ be non-empty bounded and $A \subseteq B$. Show that $\sup A \leq \sup B$.

Pf: Since A, B are non-empty and bounded by the completeness axiom, $\sup A$, $\sup B$ exist.

Common Mistakes: 1) assumed $\sup A \in A$. Consider $A = (0,1)$. Then $\sup A = 1 \notin A$.

2) $\sup A \in B$. ($A \subseteq B$). Let $A = (0,1)$. $B = (-1, 1)$. Then $A \subseteq B$, but $\sup A = 1 \notin B$.

We first show that $\sup B$ is an u.b. of A. Let $a \in A$. Then since $A \subseteq B$, $a \in B$. And since $\sup B$ is an u.b. of B, we have

$$a \leq \sup B.$$

So $\sup B$ is an u.b. of A.

Since $\sup A$ is the least upper bound of A, for any other u.b. v of A, we have $\sup A \le v$.

So take v to be $\sup B$ above, then $\sup A \le \sup B$. //

Sps $\sup A > \sup B$ Then $\varepsilon = \sup A - \sup B > 0, \ldots$

⌞ this approach will eventually work, but is also more complicated

b) Let $A \subseteq \mathbb{R}$ be bounded, nonempty, and $a \ge 0$ for all $a \in A$. Let $S = \{a^2 : a \in A\}$. Then show $\sup S = (\sup A)^2$. Does the conclusion still hold if the non-negativity assumption on A is dropped?

Basically WTS $\sup(A^2) = (\sup A)^2$.

Pf: Since A is non-empty bounded, $\sup A$ exists.

$\Downarrow$

S is also non-empty bounded. $\Rightarrow$ $\sup S$ also exists.

Common mistake: 1) $A \subseteq S$, then use part (a).

Consider $A = \{2, 5\}$. Then $S = \{4, 25\}$, and $A \not\subseteq S$.

To show $\sup S = (\sup A)^2$, we'll show $\sup S \leq (\sup A)^2$ and $(\sup A)^2 \leq \sup S$.

Let's first show $\sup S \leq (\sup A)^2$. We'll show $(\sup A)^2$ is an u.b. of S.

Let $s \in S$. Then $s = a^2$ for some $a \in A$. Since $a \leq \sup A$ for all $a \in A$,

$$s = a \cdot a \leq a \sup A \leq (\sup A)^2.$$

So $(\sup A)^2$ is an u.b. of S. So we have $\sup S \leq (\sup A)^2$.

Now we have to show $(\sup A)^2 \leq \sup S$. Suppose for the sake of contradiction that $(\sup A)^2 > \sup S$. Then $\varepsilon = \frac{(\sup A)^2 - \sup S}{2 \sup A} > 0$.

Then $\exists\, a_\varepsilon \in A$ s.t. $\sup A - \varepsilon < a_\varepsilon$.

$$\Rightarrow a_\varepsilon^2 > (\sup A - \varepsilon)^2 = (\sup A)^2 - 2 \sup A\, \varepsilon + \varepsilon^2 > (\sup A)^2 - 2 \sup A\, \varepsilon$$

$$= (\sup A)^2 - \cancel{2 \sup A} \left(\frac{(\sup A)^2 - \sup S}{\cancel{2 \sup A}} \right)$$

$$= \cancel{(\sup A)^2} - \cancel{(\sup A)^2} + \sup S$$

$\Rightarrow a_\varepsilon^2 > \sup S$, which means $\exists\, s = a_\varepsilon^2 \in S$ s.t. $s > \sup S$, a contradiction.

So $(\sup A)^2 > \sup S$ is not possible $\Rightarrow (\sup A)^2 \le \sup S$.

$$\Rightarrow \sup S = (\sup A)^2.$$

If the non-negativity assumption is dropped, the conclusion fails. For example, let $A = \{-2, -1\}$ Then $\sup A = -1$, $(\sup A)^2 = (-1)^2 = 1$.

but $S = \{1, 4\}$ and $\sup S = 4 \neq 1$.

Q4 b) Suppose $\lim_{n\to\infty} x_n = x$. Then show $\lim_{n\to\infty} \frac{x_1+x_2+\cdots x_n}{n} = x$.

["Cesaro Sums"?

Pf: Let $\varepsilon>0$ be given. Then we want to show $\exists N_\varepsilon \in \mathbb{N}$ s.t.

$$\forall n \geq N_\varepsilon, \quad \left| \frac{x_1+\cdots+x_n}{n} - x \right| < \varepsilon.$$

Common mistake:

1) Assumed $\{x_n\}$ was monotone and try to use MCT.

Consider $(x_n) = \frac{(-1)^n}{n}$.

Let's look at $\left| \frac{x_1+\cdots+x_n}{n} - x \right| = \left| \frac{x_1+\cdots+x_n - nx}{n} \right|$

$$= \left| \frac{x_1 - x + x_2 - x + \cdots + x_n - x}{n} \right|$$

Since $\lim_{n\to\infty} x_n = x$, $\exists m \in \mathbb{N}$ s.t. $\forall k \geq m$, $|x_k - x| < \frac{\varepsilon}{2}$. (1)

So $\left|\frac{x_1-x+x_2-x+\cdots+x_n-x}{n}\right| = \left|\frac{x_1-x+\cdots+x_{m-1}-x+x_m-x+\cdots+x_n-x}{n}\right|$

$$\leq \frac{1}{n}\sum_{k=1}^{m-1}|x_k-x| + \underbrace{\frac{1}{n}\sum_{k=m}^{n}|x_k-x|}$$

by (1), $\frac{1}{n}\sum_{k=m}^{n}|x_k-x| \leq \frac{1}{n}\sum_{k=m}^{n}\frac{\varepsilon}{2} = \frac{(n-m)}{n}\frac{\varepsilon}{2}$

$\leq \frac{\varepsilon}{2}$.

Since $\{x_n\}$ converges, $\{x_n\}$ is bounded.

$\Rightarrow \{x_n-x\}$ is bounded. So $\exists M\in\mathbb{R}$, s.t. $|x_n-x|\leq M$ for all n.

$$\frac{1}{n}\sum_{k=1}^{m-1}|x_k-x| \leq \frac{1}{n}\sum_{k=1}^{m-1}M = \frac{(m-1)M}{n}.$$

So we take $N_\varepsilon = \max\left\{ \frac{(m-1)M}{\varepsilon/2}, m \right\}$,

then $\left| \frac{x_1 + \cdots + x_n}{n} - x \right| \leq \underbrace{\frac{1}{n} \sum_{n=1}^{m-1} |x_n - x|}_{\leq \frac{\varepsilon}{2}} + \underbrace{\frac{1}{n} \sum_{n=m}^{n} |x_n - x|}_{\varepsilon/2}$

$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$